

# A CLASS OF NON-RATIONAL SURFACE SINGULARITIES WITH BIJECTIVE NASH MAP

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**ABSTRACT.** Let  $(\mathcal{S}, 0)$  be a germ of complex analytic normal surface. On its minimal resolution, we consider the reduced exceptional divisor  $E$  and its irreducible components  $E_i$ ,  $i \in I$ . The Nash map associates to each irreducible component  $C_k$  of the space of arcs through 0 on  $\mathcal{S}$  the unique component of  $E$  cut by the strict transform of the generic arc in  $C_k$ . Nash proved its injectivity and asked if it was bijective. As a particular case of our main theorem, we prove that this is the case if  $E \cdot E_i < 0$  for any  $i \in I$ .

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## 1. INTRODUCTION

Let  $(\mathcal{S}, 0)$  be a germ of complex analytic normal surface. Let

$$\pi_m : (\tilde{\mathcal{S}}_m, E) \rightarrow (\mathcal{S}, 0)$$

be its minimal resolution, where  $E$  is the reduced exceptional divisor of  $\pi_m$ , and let  $(E_i)_{i \in I}$  be the irreducible components of  $E$ . A resolution is called *good* if  $E$  has normal crossings and if all its components are smooth. It is important to notice that, by Grauert's contractibility theorem for surfaces [5], there exist singularities whose minimal resolution is not good.

An *arc through 0* on  $\mathcal{S}$  is a germ of formal map  $(\mathbf{C}, 0) \rightarrow (\mathcal{S}, 0)$ .

We denote by  $(\mathcal{S}, 0)_\infty$  the *space of arcs* through 0 on  $\mathcal{S}$ . It can be canonically given the structure of a scheme over  $\mathbf{C}$ , as the projective limit of schemes of finite type obtained by truncating arcs at each finite order. So it makes sense to speak about its irreducible components  $(C_k)_{k \in K}$ . For each arc represented by an element in  $C_k$ , one can consider the intersection point with  $E$  of its strict transform on  $\tilde{\mathcal{S}}_m$ . For a generic element of  $C_k$  (in the Zariski topology), this point is situated on a unique irreducible component of  $E$ . In this manner one defines a map:

$$\mathcal{N} : \{C_k \mid k \in K\} \rightarrow \{E_i \mid i \in I\}$$

which is called *the Nash map* associated to the germ  $(\mathcal{S}, 0)$ . It was defined by Nash around 1966, in a preprint published later as [17]. He proved that the map  $\mathcal{N}$  is injective (which shows in particular that  $K$  is a finite set) and asked the question:

*Is the map  $\mathcal{N}$  bijective?*

This question is now called *the Nash problem on arcs*. No germ  $(\mathcal{S}, 0)$  is known for which the answer is negative. But the bijectivity of  $\mathcal{N}$  was only proved till now for special classes of singularities:

- for the germs of type  $(\mathbf{A}_n)_{n \geq 1}$  by Nash himself in [17];
- for normal minimal singularities by Reguera [21]; different proofs were given by Plénat [19] and by Fernández-Sánchez [4];

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- for sandwiched singularities it was sketched by Reguera [22], using her common work [15] with Lejeune-Jalabert on the wedge problem.
- for the germs of type  $(\mathbf{D}_n)_{n \geq 4}$  by Plénat [19];
- for the germs with a good  $\mathbf{C}^*$ -action such that the curve  $\text{Proj } \mathcal{S}$  is not rational, it follows immediately by combining results of Lejeune-Jalabert [13] and Reguera [22].

With the exception of the last class, all the other ones consist only in *rational* singularities and can be defined purely topologically.

Here we prove that the Nash map is bijective for a new class of surface singularities (Theorem 5.1), whose definition depends only on the intersection matrix of the minimal resolution and not on the genera or possible singularities of the components  $E_i$ . In particular, their minimal resolution may not be good, which contrasts with the classes of singularities described before. The following (Corollary 5.2) is a particular case of the main theorem:

*If  $E \cdot E_i < 0$  for any  $i \in I$ , then the Nash map  $\mathcal{N}$  is bijective.*

We also show (Corollary 5.5) that the hypothesis of the previous corollary are verified by an infinity of topologically pairwise distinct *non-rational* singularities, which explains the title of the article.

The Nash map can also be defined in higher dimensions, over any field, for not necessarily normal schemes which admit resolutions of their singularities. It is always injective and the same question can also be asked. Ishii and Kollar proved in [6] (a good source for everything we use about spaces of arcs and the Nash map, as well as for references on related works) that it is not always bijective. Indeed, they gave a counterexample in dimension 4, which can be immediately transformed in a counterexample in any larger dimension. They left open the cases of dimensions 2 and 3...

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## 2. A CRITERION FOR DISTINGUISHING COMPONENTS OF THE SPACE OF ARCS

Consider a germ  $(\mathcal{S}, 0)$  of normal surface and its minimal resolution morphism  $\pi_m : (\tilde{\mathcal{S}}_m, E) \rightarrow (\mathcal{S}, 0)$ . If  $D$  is a divisor on  $\tilde{\mathcal{S}}_m$ , it can be uniquely written as the sum of a divisor supported by  $E$  - called the *exceptional part* of  $D$  - and a divisor whose support meets  $E$  in a finite number of points. If  $D$  consists only of its exceptional part, we say that  $D$  is *purely exceptional*.

For each  $i \in I$ , let  $v_{E_i}$  be the divisorial valuation defined by  $E_i$  on the fraction field of the analytic local ring  $\mathcal{O}_{\mathcal{S}, 0}$ . Denote by  $\mathfrak{m}_{\mathcal{S}, 0}$  the maximal ideal of this local ring. If  $f \in \mathfrak{m}_{\mathcal{S}, 0}$ , the exceptional part of  $\text{div}(f \circ \pi_m)$  is precisely  $\sum_{i \in I} v_{E_i}(f)E_i$ .

For each component  $E_i$  of  $E$ , consider the arcs on  $\tilde{\mathcal{S}}_m$  whose closed points are on  $E_i - \cup_{j \neq i} E_j$  and which intersect  $E_i$  transversally. Consider the set of their images in  $(\mathcal{S}, 0)_\infty$  and denote its closure by  $V(E_i)$ . The sets  $V(E_i)$  are irreducible and

$$(\mathcal{S}, 0)_\infty = \bigcup_{i \in I} V(E_i)$$

(see Lejeune-Jalabert [14, Appendix 3]).

The following proposition is a special case of a general one proved by the first author in [18] (see also [19]) for non-necessarily normal germs of any dimension and

for arbitrary resolutions. It generalizes an equivalent result proved by Reguera [21, Theorem 1.10] for the case of rational surface singularities.

**Proposition 2.1.** *If there exists a function  $f \in \mathfrak{m}_{\mathcal{S},0}$  such that  $v_{E_i}(f) < v_{E_j}(f)$ , then  $V(E_i) \not\subseteq V(E_j)$ .*

*Proof:* Let  $(\mathcal{S}, 0) \hookrightarrow (\mathbf{C}^n, 0)$  be an analytic embedding of the germ  $(\mathcal{S}, 0)$ . Denote by  $(x_1, \dots, x_n)$  the coordinates of  $\mathbf{C}^n$ . An arc  $\phi \in (\mathcal{S}, 0)_\infty$  is then represented by  $n$  formal power series  $(x_k(t) = \sum_{l=1}^{\infty} a_{kl} t^l)_{1 \leq k \leq n}$ , where the coefficients  $(a_{kl})_{k,l}$  are subjected to algebraic constraints, coming from the fact that the arc must lie on  $\mathcal{S}$ .

For each  $j \in I$ , a Zariski open set  $U_f(E_j)$  in  $V(E_j)$  consists of the images by  $\pi_m$  of the arcs on  $\tilde{\mathcal{S}}_m$  which meet transversely  $E_j$  in a smooth point of  $\text{div}(f \circ \pi_m)$ . If  $\phi \in U_f(E_j)$ , we have:

$$v_{E_j}(f) = v_t(f \circ \phi)$$

where  $v_t(g)$  denotes the order in  $t$  of  $g \in \mathbf{C}[[t]]$ .

This shows that the first  $(v_{E_j}(f) - 1)$  coefficients of  $f \circ \phi$ , seen as elements of  $\mathbf{C}[a_{kl}]_{k,l}$ , must vanish. Their vanishing defines a closed subscheme  $Z_{f,j}$  of  $(\mathcal{S}, 0)_\infty$ . Therefore,  $U_f(E_j) \subset Z_{f,j}$ , which implies that:

$$V(E_j) \subset Z_{f,j}.$$

As  $v_{E_i}(f) < v_{E_j}(f)$ , we see that no element of  $U_f(E_i)$  is included in  $Z_{f,j}$ , which shows that:

$$V(E_i) \not\subseteq Z_{f,j}.$$

The proposition follows.  $\square$

### 3. CONSTRUCTION OF FUNCTIONS WITH PRESCRIBED DIVISOR

In this section,  $\pi : (\tilde{\mathcal{S}}, E) \rightarrow (\mathcal{S}, 0)$  denotes *any* resolution of  $(\mathcal{S}, 0)$ .

Inside the free abelian group generated by  $(E_i)_{i \in I}$  we consider the set:

$$\mathcal{L}(\pi) := \{D \mid D \neq 0, D \cdot E_i \leq 0, \forall i \in I\}.$$

It is a semigroup with respect to addition, which we call (following Lê [12, 3.2.5]) *the Lipman semigroup* associated to  $\pi$  (see Lipman [16, §18]). It is known that it consists only of effective divisors (see Lipman [16, §18 (ii)]).

We call *strict Lipman semigroup* of  $\pi$  the subset:

$$\mathcal{L}^\circ(\pi) := \{D \in \mathcal{L}(\pi) \mid D \cdot E_i < 0, \forall i \in I\}$$

of the Lipman semigroup of  $\pi$ . It is always non-empty.

The importance of the Lipman semigroup comes from the fact that the exceptional parts of the divisors of the form  $\text{div}(f \circ \pi)$ , where  $f \in \mathfrak{m}_{\mathcal{S},0}$ , are elements of it. The converse is true for rational surface singularities, but this is not the case for arbitrary surface singularities.

We give now a numerical criterion on a divisor  $D \in \mathcal{L}(\pi)$  which allows one to conclude that it is the exceptional part of a divisor of the form  $\text{div}(f \circ \pi)$ :

**Proposition 3.1.** *Let  $D$  be an effective purely exceptional divisor, such that for any  $i, j \in I$ , one has the inequality:*

$$(D + E_i + K_{\tilde{\mathcal{S}}}) \cdot E_j + 2\delta_i^j \leq 0$$

where  $\delta_i^j$  is Kronecker's symbol. Then there exists a function  $f \in \mathfrak{m}_{\mathcal{S},0}$  such that the exceptional part of  $\text{div}(f \circ \pi)$  is precisely  $D$ .

*Proof:* We use the following Grauert-Riemenschneider type vanishing theorem, proved by Laufer [9] for analytic germs and by Ramanujam [20] for algebraic ones (see also Bădescu [1, 4.1]):

If  $L$  is a divisor on  $\tilde{\mathcal{S}}$  such that  $L \cdot E_j \geq K_{\tilde{\mathcal{S}}} \cdot E_j$ ,  $\forall j \in I$ , then  $H^1(\mathcal{O}_{\tilde{\mathcal{S}}}(L)) = 0$ .

We apply the theorem to  $L = -D - E_i$ , for any  $i \in I$ . Our hypothesis implies that  $H^1(\mathcal{O}_{\tilde{\mathcal{S}}}(-D - E_i)) = 0$ . Then, from the exact cohomology sequence associated to the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\tilde{\mathcal{S}}}(-D - E_i) \longrightarrow \mathcal{O}_{\tilde{\mathcal{S}}}(-D) \xrightarrow{\psi_i} \mathcal{O}_{E_i}(-D) \longrightarrow 0$$

we deduce the *surjectivity* of the restriction map

$$\psi_{i*} : H^0(\mathcal{O}_{\tilde{\mathcal{S}}}(-D)) \rightarrow H^0(\mathcal{O}_{E_i}(-D))$$

By Serre duality on the irreducible (possibly singular) curve  $E_i$  (see Reid [23, 4.10]), we get:

$$h^1(\mathcal{O}_{E_i}(-D)) = h^0(\mathcal{O}_{E_i}(K_{\tilde{\mathcal{S}}} + E_i + D)) = 0$$

For the last equality we have used the hypothesis  $(D + E_i + K_{\tilde{\mathcal{S}}}) \cdot E_i \leq -2 < 0$ , which shows that the line bundle  $\mathcal{O}_{E_i}(K_{\tilde{\mathcal{S}}} + E_i + D)$  cannot have a non-trivial section.

By applying the Riemann-Roch theorem and the adjunction formula for the irreducible curve  $E_i$  of arithmetic genus  $p_a(E_i)$  (see Reid [23, 4.11]), we get:

$$\begin{aligned} h^0(\mathcal{O}_{E_i}(-D)) &= h^0(\mathcal{O}_{E_i}(-D)) - h^1(\mathcal{O}_{E_i}(-D)) = \\ &= \chi(\mathcal{O}_{E_i}(-D)) = \\ &= 1 - p_a(E_i) - D \cdot E_i \geq \\ &\geq 1 - p_a(E_i) + (K_{\tilde{\mathcal{S}}} + E_i) \cdot E_i + 2 = \\ &= 1 - p_a(E_i) + 2p_a(E_i) - 2 + 2 = \\ &= 1 + p_a(E_i) > 0 \end{aligned}$$

This shows that there exists a non-identically zero section  $s_i \in H^0(\mathcal{O}_{E_i}(-D))$ . The surjectivity of  $\psi_{i*}$  implies that there exists  $\sigma_i \in H^0(\mathcal{O}_{\tilde{\mathcal{S}}}(-D))$  such that  $\psi_{i*}(\sigma_i) = s_i$ . As  $(\mathcal{S}, 0)$  is normal, there exists  $f_i \in \mathfrak{m}_{\mathcal{S}, 0}$  with  $\sigma_i = f_i \circ \pi$ . If we write  $D = \sum_{j \in I} a_j E_j$ , we see that  $v_{E_i}(f_i) = a_i$  and  $v_{E_j}(f_i) \geq a_j$ , for all  $j \neq i$ . We deduce that any generic linear combination  $f = \sum_{i \in I} \lambda_i f_i$  of the functions so constructed verifies:

$$v_{E_i}(f) = a_i, \quad \forall i \in I.$$

The proposition is proved. □

**Remark 3.2. a)** The proof follows the same line as the one of Proposition 3.1 of [2] and 4.1 of [3], proved by Caubel, Némethi and Popescu-Pampu. The difference here is that we no longer deal with a good resolution of  $(\mathcal{S}, 0)$  and we do not ask for a precise knowledge of the topological type of the total transform of  $f$ .

**b)** As Morales informed us, Laufer [10, 3.1] proved a related statement when  $\pi$  is the minimal resolution of  $(\mathcal{S}, 0)$ : if  $L$  is a line bundle on  $\tilde{\mathcal{S}}_m$  with  $L \cdot E_i \geq 2K_{\tilde{\mathcal{S}}_m} \cdot E_i$ ,  $\forall i \in I$ , then  $L$  has no base points on  $M$ . This implies immediately, by taking  $L = \mathcal{O}_{\tilde{\mathcal{S}}_m}(-D)$ , that the conclusion of Proposition 3.1 is true with the hypothesis  $(D + 2K_{\tilde{\mathcal{S}}_m}) \cdot E_i \leq 0$ ,  $\forall i \in I$ . As in the sequel we work only with the minimal resolution, we could have chosen to use Laufer's criterion. By looking at the way we use Proposition 3.1 in the proof of Proposition 4.3, one sees that this would have been enough in order to prove our main Theorem 5.1. Nevertheless, we think that Proposition 3.1 has independent interest. Indeed, neither Laufer's hypothesis nor ours implies the other one. In order to see it, consider first a singularity whose minimal resolution has an irreducible exceptional divisor  $E$  which is smooth, of genus 2 and with  $E^2 = -1$ : then

$D = 4E$  verifies our hypothesis, but not Laufer's. Secondly, by considering another singularity with  $E = E_1 + E_2$ ,  $p_a(E_1) = p_a(E_2) = 0$ ,  $E_1^2 = -4$ ,  $E_2^2 = -2$ ,  $E_1 \cdot E_2 = 2$ , one sees that  $D = 4E_1 + 4E_2$  verifies Laufer's hypothesis but not ours.

#### 4. THE CONDITIONS (\*) AND (\*\*)

From now on, we deal again with the minimal resolution  $\pi_m$  of the germ  $(\mathcal{S}, 0)$ . Inside the real vector space with basis  $(E_i)_{i \in I}$ , we consider the open half-spaces:

$$\left\{ \sum_{i \in I} a_i E_i \mid a_i < a_j \right\},$$

for each  $i \neq j$ . We call them *the fundamental half-spaces* of  $\pi_m$ .

We introduce two conditions on the normal singularity  $(\mathcal{S}, 0)$ :

- (\*) *the intersection of  $\mathcal{L}^\circ(\pi_m)$  with each fundamental half-space is non-empty;*
- (\*\*)  $E \in \mathcal{L}^\circ(\pi_m)$ .

Notice that both conditions depend only on the intersection matrix of  $E$ . In particular, they are purely topological.

**Lemma 4.1.** *The germs which verify condition (\*\*) form a strict subset of those which verify condition (\*).*

*Proof:* Let  $(\mathcal{S}, 0)$  verify condition (\*\*). Then, for  $n \in \mathbb{N}^*$  big enough (in fact for  $n > \max\{\frac{E_i \cdot E_j}{|E \cdot E_i|}\}$ ), one sees from the definition of the strict Lipman semigroup that  $nE + E_j \in \mathcal{L}^\circ(\pi_m)$ ,  $\forall j \in I$ . But each fundamental half-space contains at least one of the divisors  $nE + E_j$ , which shows that  $(\mathcal{S}, 0)$  verifies condition (\*).

Consider then any normal singularity  $(\mathcal{S}, 0)$  whose minimal resolution has the same intersection matrix as a singularity of type  $\mathbf{A}_n$  (that is,  $E = \sum_{i=1}^n E_i$  with  $E_i^2 = -2$ ,  $\forall i \in \{1, \dots, n\}$ ,  $E_i \cdot E_{i+1} = 1$ ,  $\forall i \in \{1, \dots, n-1\}$  and  $E_i \cdot E_j = 0$ ,  $\forall i, j \in \{1, \dots, n\}$  such that  $|i - j| \notin \{0, 1\}$ ), the components  $E_i$  having otherwise arbitrary genera and singularities. Take  $n \geq 3$ . Then  $E \cdot E_i = 0$ ,  $\forall i \in \{2, \dots, n-1\}$ , which shows that  $E \notin \mathcal{L}^\circ(\pi_m)$ . That is, the germ  $(\mathcal{S}, 0)$  does not verify condition (\*\*).

Define

$$\alpha_k := nk - \frac{(k-1)k}{2}$$

for any  $k \in \{1, \dots, n\}$ . Then it is immediate to see from the definitions that the divisors  $D := \sum_{k=1}^n \alpha_k E_k$  and  $D' := \sum_{k=1}^n \alpha_{n+1-k} E_k$  are in the strict Lipman semigroup and that each fundamental half-space contains exactly one of them. Therefore, the germ  $(\mathcal{S}, 0)$  verifies condition (\*). This shows that the inclusion stated in the Lemma is strict.  $\square$

**Remark 4.2. a)** *We choose to distinguish inside the class of singularities which verify condition (\*) those which verify condition (\*\*) for computational convenience, because this second condition is more readily verified on a given intersection matrix.*

**b)**  *$\mathbf{A}_n$ -type singularities are the only rational double points which verify condition (\*). This illustrates the difficulty of dealing with  $\mathbf{D}_n$ -type singularities (see Plénat [19]).*

The motivation to introduce condition (\*) comes from the following proposition:

**Proposition 4.3.** *Suppose that  $(\mathcal{S}, 0)$  verifies condition (\*). Then, for any pair of distinct indices  $i, j \in I$ , there exists a function  $f \in \mathfrak{m}_{\mathcal{S}, 0}$  such that  $v_{E_i}(f) < v_{E_j}(f)$ .*

*Proof:* Let  $i, j \in I$  satisfy  $i \neq j$ . As  $(\mathcal{S}, 0)$  verifies condition (\*), there exists  $D = \sum_{l \in I} a_l E_l \in \mathcal{L}^\circ(\pi_m)$  such that  $a_i < a_j$ . Then, for  $n \in \mathbf{N}^*$  big enough, one has  $(nD + E_k + K_{\tilde{\mathcal{S}}}).E_l + 2\delta_k^l \leq 0, \forall k, l \in I$ . By Proposition 3.1, there exists  $f \in \mathfrak{m}_{\mathcal{S}, 0}$  such that  $\text{div}(f \circ \pi_m) = nD$ , which shows that  $v_{E_i}(f) = na_i < na_j = v_{E_j}(f)$ .  $\square$

**Remark 4.4.** *The referee suggests us the following alternative proof of the previous proposition : let  $D = \sum_{l \in I} a_l E_l \in \mathcal{L}^\circ(\pi_m)$  such that  $a_i < a_j$ . Then  $\mathcal{O}_{\tilde{\mathcal{S}}_m}(-D)$  is ample (see Lipman [16, 10.4 and proof of 12.1 (iii)]). Thus there exists  $n \in \mathbf{N}^*$  such that  $\mathcal{O}_{\tilde{\mathcal{S}}_m}(-nD)$  is generated by its global sections. Therefore there exists  $f \in \mathfrak{m}_{\mathcal{S}, 0}$  such that the exceptional part of  $\text{div}(f \circ \pi)$  is precisely  $nD$ . For such an  $f$ , we have of course  $v_{E_i}(f) < v_{E_j}(f)$ .*

## 5. THE PROOF OF THE MAIN THEOREM

Our main theorem is:

**Theorem 5.1.** *Suppose that  $(\mathcal{S}, 0)$  verifies condition (\*). Then the Nash map  $\mathcal{N}$  is bijective.*

*Proof:* By combining Proposition 2.1 and Proposition 4.3, we deduce that  $V(E_i) \not\subseteq V(E_j)$  for any  $i \neq j$ . As  $(\mathcal{S}, 0)_\infty = \bigcup_{i \in I} V(E_i)$ , we deduce that the schemes  $(V(E_i))_{i \in I}$  are precisely the irreducible components  $(C_k)_{k \in K}$  of  $(\mathcal{S}, 0)_\infty$ . As the Nash map  $\mathcal{N}$  is injective, this shows its surjectivity.

The Theorem is proved.  $\square$

Using Lemma 4.1, we get as an immediate corollary:

**Corollary 5.2.** *Suppose that  $(\mathcal{S}, 0)$  verifies condition (\*\*). Then the Nash map  $\mathcal{N}$  is bijective.*

We denote by  $\Gamma(E)$  the *dual graph* of  $E$ , whose vertices correspond bijectively to the components  $(E_i)_{i \in I}$ , the vertex  $E_i$  being weighted by  $|E_i|^2$  and the vertices  $E_i, E_j$  being joined by  $E_i.E_j$  vertices, for any  $i \neq j$ . Let  $\gamma(E_i)$  denote the number of edges which start from the vertex  $E_i$  (so, each loop based at the vertex  $E_i$  counts for 2).

The next proposition characterizes rational singularities among those which verify condition (\*\*).

**Proposition 5.3.** *A singularity which verifies condition (\*\*) is rational if and only if the following conditions are simultaneously verified:*

- (i)  $\Gamma(E)$  is a tree;
- (ii)  $E_i \simeq \mathbf{P}^1, \forall i \in I$ ;
- (iii)  $|E_i|^2 > \gamma(E_i), \forall i \in I$ .

*Proof:* First of all, notice that conditions (i) and (ii) imply that  $\pi_m$  is a good resolution.

Suppose that  $(\mathcal{S}, 0)$  is rational and verifies condition (\*\*). Then conditions (i) and (ii) are verified, as general properties of rational singularities (see Bădescu [1, 3.32.3]). As  $E.E_i = -|E_i|^2 + \gamma(E_i)$ , condition (iii) is also verified.

Conversely, suppose that the conditions (i), (ii) and (iii) are verified. By a result of Spivakovskiy [24, II] (see also Lê [11, 5.3]), this shows that  $(\mathcal{S}, 0)$  is a normal minimal singularity, and in particular it is rational. Consult the references above for the notion of minimal surface singularity, as well as Kollar [7], where this notion was introduced in arbitrary dimensions.  $\square$

**Remark 5.4.** a) Normal minimal surface singularities are precisely those whose minimal good resolution verifies conditions (i), (ii) and (iii)':  $|E_i^2| \geq \gamma(E_i)$ ,  $\forall i \in I$ . The stronger condition (iii) is equivalent to the fact that the minimal resolution can be obtained by blowing-up once the origin (see Spivakovsky [24, II], Lê [11, 6.1]): one says that the singularity is superisolated. Thus, Proposition 5.3 is equivalent to the fact that the rational surface singularities which verify condition (\*\*) are precisely the superisolated minimal ones (a remark we owe to Lejeune-Jalabert).

b) There are rational singularities which verify condition (\*) but do not verify condition (\*\*). Consider for example a germ of normal surface whose minimal resolution  $\pi_m$  is good and has a reduced exceptional divisor  $E$  with four components of genus 0 such that  $E_1^2 = E_2^2 = E_3^2 = -n \leq -5$ ,  $E_4^2 = -2$ ,  $E_1E_4 = E_2E_4 = E_3E_4 = 1$ ,  $E_1E_2 = E_2E_3 = E_3E_1 = 0$ . Then it is immediate to verify that the divisors:

$$(2n+1)(E_1 + E_2 + E_3) + 4nE_4 \\ (2n^2 - 2n + 3)E_1 + 3n(E_2 + E_3) + (n^2 + 3n)E_4$$

as well as those obtained by permuting  $E_1, E_2, E_3$  are in  $\mathcal{L}^\circ(\pi_m)$  and that each fundamental half-space contains at least one of them. So, the germ verifies condition (\*). But, as  $E \cdot E_4 > 0$ , the germ does not verify condition (\*\*). Moreover, the dual graph  $\Gamma(E)$  is a subgraph of the dual graph associated to the resolution of a plane curve singularity (attach  $n-1$  vertices to  $E_1$ ,  $n-2$  vertices to  $E_2$  and  $n-3$  vertices to  $E_3$ , all of them weighted by -1). Using Lê [11, 4.8] (see also Spivakovsky [24, II]), we see that the singularity is sandwiched, and in particular it is rational.

As condition (iii) in the Proposition 5.3 is equivalent with condition (\*\*), we see immediately that any (abstract) graph is the dual graph of the minimal resolution of a singularity which verifies condition (\*\*), once the weights of the vertices are negative enough. Then, if one of the conditions (i) or (ii) is not satisfied, we are in presence of a non-rational singularity. This shows:

**Corollary 5.5.** *There exists an infinity of pairwise topologically distinct normal non-rational surface singularities which verify condition (\*\*), and consequently for which the Nash map is bijective.*

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